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Parastrophic-orthogonal ternary medial quasigroups with 3 and 4 distinct parastrophes

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Abstract. In the article, we study parastrophic-orthogonal ternary quasigroups: namely, group isotopes which have 3 and 4 distinct parastrophes. The necessary and sufficient conditions for ternary medial quasigroups with 3 and 4 distinct parastrophes to be totally parastrophic-orthogonal are proved. The conditions under which these quasigroups are strongly parastrophic-orthogonal are described. Thus, some methods of constructing orthogonal and strongly orthogonal ternary quasigroups are obtained.

Keywords: ternary quasigroup, group isotope, medial quasigroup, parastrophe, (strongly) orthogonal quasigroups, totally parastrophic-orthogonal (top) quasigroup.

1. Introduction

Ternary quasigroups possessing a certain number of pairwise distinct parastrophes and their existence were studied in M. McLeish's papers [1] and [2]. Later F. Sokhatsky and Ye. Pirus in [3] and [4] described a classification and canonical decompositions of ternary group isotopes possessing various numbers of distinct parastrophes. The study of ternary quasigroups with orthogonal parastrophes is a natural completion of these results. The conditions for a medial ternary quasigroup to be totally self-orthogonal (i.e. all distinct

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principal parastrophes are orthogonal) were proved in [5] for the case when all its principal parastrophes are pairwise different. This approach was proposed by G. Belyavskaya and T. Rotari (Popovich) in [6] who described the conditions for a central binary quasigroup to be totally parastrophic-orthogonal.

Each parastrophe of an invertible operation can be regarded as a principal operation, and the regularities that arise in the study of these quasigroups are expressed in terms of parastrophic symmetry. Suppose that f is an n-ary invertible operation and ${}^{\sigma}f$ denotes a σ-parastrophe of f, $\sigma \in S_n$. The mapping $(\sigma, f) \mapsto {}^{\sigma}f$ is an action of the symmetric group S_{n+1} on the set of all invertible n-ary operations defined on a carrier and is called a parastrophic action [7]. The stabilizer group $\mathrm{Ps}(f)$ and the orbit $\mathrm{Po}(f)$ of an operation f

$$\mathrm{Ps}(f) := \{\sigma \in S_{n+1} \mid {}^{\sigma}f = f\} \leqslant S_{n+1}, \qquad \mathrm{Po}(f) := \{{}^{\sigma}f \mid \sigma \in S_{n+1}\}$$

are called a *parastrophic symmetry group* and a *parastrophic orbit* of the operation f respectively. The series of statements follows from classical group theory, specifically

$$|\mathrm{Ps}(f)| \cdot |\mathrm{Po}(f)| = (n+1)!, \qquad \mathrm{Ps}({}^{\sigma}f) = \sigma \mathrm{Ps}(f) \sigma^{-1}.$$

Therefore, parastrophic quasigroups belong to the same parastrophic orbit and their parastrophic symmetry groups are conjugated.

2. Statement of the problem and preliminaries

We restrict our attention to the symmetric group S_4. It is known fact that S_4 has 30 subgroups distributed into 11 conjugacy classes. Here, we consider groups of parastrophic symmetry D_8, S_3 and A_3, where

$$\begin{aligned} D_8 &:= \{\iota,\ (12),\ (34),\ (12)(34),\ (13)(24),\ (14)(23),\ (1324),\ (1423)\}, \\ S_3 &:= \{\iota,\ (12),\ (13),\ (23),\ (123),\ (132)\}, \qquad A_3 := \{\iota,\ (123),\ (132)\}. \end{aligned}$$

In the article, we consider the conditions when a ternary medial quasigroup with the parastrophic symmetry group D_8, S_3 and A_3 possesses the property that it is parastrophic-orthogonal or totally parastrophic-orthogonal, i.e., in the cases when the quasigroup has 3 and 4 pairwise distinct parastrophes. Before proceeding further, we need the following definitions and statements.

Throughout the article, all operations are defined on a fixed set Q called a *carrier set* or a *carrier* and $m := |Q| < \infty$. We will often use the following lemma.

Lemma 1. *A product of elements in a finite monoid is invertible if and only if each of these elements is invertible.*

A ternary operation f defined on Q is called *invertible* or a *quasigroup operation* and the pair $(Q; f)$ is called a *quasigroup* of order m, if each of the terms $f(x, a, b)$, $f(a, x, b)$, $f(a, b, x)$ defines a permutation of Q for all $a, b \in Q$.

Orthogonality. A triplet of ternary operations f_1, f_2, f_3 is called *orthogonal*, if the system of equations

$$\begin{cases} f_1(x_1, x_2, x_3) = a_1, \\ f_2(x_1, x_2, x_3) = a_2, \\ f_3(x_1, x_2, x_3) = a_3 \end{cases}$$

has a unique solution for all $a_1, a_2, a_3 \in Q$. A set of ternary operations $\Sigma = \{f_1, f_2, \dots, f_s\}$, $s \geqslant 3$, is called *orthogonal*, if each triplet of distinct operations from Σ is orthogonal. Operations f_1, f_2, f_3 are called *strongly orthogonal* if the set of operations $\{f_1, f_2, f_3, e_1, e_2, e_3\}$ is orthogonal, where e_i defined by the equality

$$e_i(x_1, x_2, x_3) = x_i$$

is called an *i-th selector*, $i \in \{1, 2, 3\}$.

The operation αf defined by

$$(\alpha f)(x, y, z) := \alpha\left(f(x, y, z)\right),$$

where α is a permutation of Q, is called a *torsion* of f.

Proposition 2. [5, Proposition 1] *If a set of operations is orthogonal, then their torsions are also orthogonal.*

Parastrophes and parastrophic symmetry. For each permutation $\sigma \in S_4$, a *σ-parastrophe* ${}^{\sigma}f$ of an invertible ternary operation f is defined by

$${}^{\sigma}f(x_{1\sigma}, x_{2\sigma}, x_{3\sigma}) = x_{4\sigma} :\Longleftrightarrow f(x_1, x_2, x_3) = x_4,$$

which is equivalent to

$${}^{\sigma}f(x_1, x_2, x_3) = x_4 :\Longleftrightarrow f(x_{1\sigma^{-1}}, x_{2\sigma^{-1}}, x_{3\sigma^{-1}}) = x_{4\sigma^{-1}}. \tag{1}$$

For all permutations $\sigma, \tau \in S_4$ and for each invertible operation f, it is true that

$${}^{\sigma}({}^{\tau}f) = {}^{\sigma\tau}f, \qquad {}^{\iota}f = f. \tag{2}$$

A σ-parastrophe is called:

- an *i-th division* if $\sigma = (i4)$ for $i = 1, 2, 3$;
- a *principal parastrophe* if $4\sigma = 4$.

The formula (1) implies that any principal σ-parastrophe can be defined by

$${}^{\sigma}f(x_1, x_2, x_3) = f(x_{1\sigma^{-1}}, x_{2\sigma^{-1}}, x_{3\sigma^{-1}}).$$

Each ternary invertible operation has $4! = 24$ parastrophes including $3! = 6$ principal parastrophes.

Theorem 3. [3, 7] *The relations (2) imply that the symmetric group S_4 acts on the set Δ_3 of all ternary invertible operations defined on a set Q. Therefore, the following statements are true:*

(1) *parastrophy is an equivalence relation on Δ_3; each block under the action is a set of all pairwise parastrophic operations, where f is one of these operation (representative);*
(2) $\mathrm{Ps}(f)$ *is a subgroup of S_4;*
(3) *parastrophic symmetry groups of parastrophic operations are isomorphic, i.e., they are conjugated: namely, $\mathrm{Ps}({}^{\sigma}f) = \sigma \mathrm{Ps}(f) \sigma^{-1}$;*
(4) *the number of all different parastrophes of f equals $\dfrac{24}{|\mathrm{Ps}(f)|}$;*
(5) *parastrophes ${}^{\sigma}f$ and ${}^{\tau}f$ are different if and only if σ and τ belong to different elements of the set $S_4/\mathrm{Ps}(f)$.*

$\mathfrak{P}(H)$ denotes the class of all quasigroups whose parastrophic symmetry group includes the subgroup $H \leqslant S_4$. Note that $\mathfrak{P}(H)$ is a variety [3].

By item 5 of Theorem 3, all distinct parastrophes of a quasigroup with the groups of parastrophic symmetry D_8, S_3 or A_3 are representatives from the elements of the sets

$$S_4/D_8 = \{D_8,\ (23)D_8,\ (13)D_8\}, \qquad S_4/S_3 = \{S_3,\ (14)S_3,\ (24)S_3,\ (34)S_3\},$$
$$S_4/A_3 = \{A_3,\ (12)A_3,\ (14)A_3,\ (24)A_3,\ (34)A_3,\ (124)A_3,\ (134)A_3,\ (142)A_3\}.$$

A ternary quasigroup is called:

- *parastrophic-orthogonal* if it has a triplet of orthogonal parastrophes;
- *self-orthogonal* if it has a triplet of orthogonal principal parastrophes;
- *totally parastrophic orthogonal* (briefly, *a top quasigroup*) if its all distinct parastrophes are orthogonal.

Group isotopes. A ternary groupoid $(Q; f)$ is called a *group isotope*, if there exists a group $(G; \cdot)$ and bijections α, β, γ from Q to G such that

$$f(x, y, z) = \delta^{-1}(\alpha x \cdot \beta y \cdot \gamma z) \qquad \forall x, y, z \in Q.$$

Each group isotope $(Q; f)$ has a 0-*canonical decomposition* $(+, \alpha_1, \alpha_2, \alpha_3, a)$ for every element $0 \in Q$, i.e.,

$$f(x_1, x_2, x_3) = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + a, \tag{3}$$

for some group $(Q; +, 0)$, permutations α_1, α_2, α_3 with $\alpha_1 0 = \alpha_2 0 = \alpha_3 0$ and $a \in Q$; $(Q; +, 0)$ is called 0-*canonical decomposition group*.

If a ternary quasigroup $(Q; f)$ is *linear* over a group $(Q; +)$, then it has decomposition (3), where $\alpha_1, \alpha_2, \alpha_3$ are automorphisms of $(Q; +)$ and $a \in Q$. If $(Q; +)$ is abelian, then $(Q; f)$ is called a *central* or *T-quasigroup*.

Corollary 4. [8] *A quasigroup $(Q; f)$ is medial if and only if there exists an abelian group $(Q; +)$ such that (3), where α_1, α_2, α_3 are pairwise commuting automorphisms of $(Q; +)$ and $a \in Q$.*

All parastrophes of a group isotope can be obtained by the following lemma.

Lemma 5. *Let $(Q; f)$ be an arbitrary ternary group isotope and let (3) be its canonical decomposition. Then its divisions and principal parastrophes are*

$$\begin{aligned}
{}^{(14)}f(x_1, x_2, x_3) &= \alpha_1^{-1}\left(x_1 - a - \alpha_3 x_3 - \alpha_2 x_2\right),\\
{}^{(24)}f(x_1, x_2, x_3) &= \alpha_2^{-1}\left(-\alpha_1 x_1 + x_2 - a - \alpha_3 x_3\right),\\
{}^{(34)}f(x_1, x_2, x_3) &= \alpha_3^{-1}\left(-\alpha_2 x_2 - \alpha_1 x_1 + x_3 - a\right),\\
{}^{\sigma}f(x_1, x_2, x_3) &= \alpha_1 x_{1\sigma^{-1}} + \alpha_2 x_{2\sigma^{-1}} + \alpha_3 x_{3\sigma^{-1}} + a, \qquad \sigma \in S_3.
\end{aligned}$$

Lemma 6. *Let $(Q; f)$ be a medial ternary quasigroup $(Q; f)$ with (3), τ_1, τ_2, $\tau_3 \in S_4$. The parastrophes ${}^{\tau_1}f$, ${}^{\tau_2}f$, ${}^{\tau_3}f$ are*

(1) *orthogonal if and only if the determinant*

$$\begin{vmatrix}
\alpha_{1\tau_1} & \alpha_{2\tau_1} & \alpha_{3\tau_1}\\
\alpha_{1\tau_2} & \alpha_{2\tau_2} & \alpha_{3\tau_2}\\
\alpha_{1\tau_3} & \alpha_{2\tau_3} & \alpha_{3\tau_3}
\end{vmatrix} \tag{4}$$

is an automorphism of the group $(Q;+)$*, where* $\alpha_4 := -\iota$*;*

(2) *strongly orthogonal if and only if the determinant* (4) *and all its minors are automorphisms of the group* $(Q;+)$.

Proof. The first statement is taken from [5].

The parastrophes ${}^{\tau_1}f$, ${}^{\tau_2}f$, ${}^{\tau_3}f$ are strongly orthogonal if and only if ${}^{\tau_1}f$, ${}^{\tau_2}f$, ${}^{\tau_3}f$, e_1, e_2, e_3 are orthogonal, i.e., each triplet of this set is orthogonal. By item 1, orthogonality of these parastrophes is equivalent to invertibility of (4), and clearly e_1, e_2, e_3 are always orthogonal. Now, consider the cases when one of the operations is a selector, say the triplet ${}^{\tau_i}f$, ${}^{\tau_j}f$, e_1 for all $i, j \in \{1, 2, 3\}$ and $i \neq j$. Then its orthogonality is equivalent to invertibility of the determinant

$$\begin{vmatrix} \alpha_{1\tau_i} & \alpha_{2\tau_i} & \alpha_{3\tau_i} \\ \alpha_{1\tau_j} & \alpha_{2\tau_j} & \alpha_{3\tau_j} \\ \iota & 0 & 0 \end{vmatrix} = \begin{vmatrix} \alpha_{2\tau_i} & \alpha_{3\tau_i} \\ \alpha_{2\tau_j} & \alpha_{3\tau_j} \end{vmatrix}.$$

Hence, orthogonality of ${}^{\tau_i}f$, ${}^{\tau_j}f$, e_1 is equivalent to invertibility of obtained minor. The proof is similar for the selectors e_2 and e_3. Thus, we get all minors of (4). The result of the lemma follows. □

Canonical decompositions of group isotopes. Let $(Q; f)$ be a ternary quasigroup with $\mathrm{Ps}(f) = D_8$. By Proposition 4 from [3], only different parastrophes of f are f, ${}^{(14)}f$, ${}^{(24)}f$.

Theorem 7. [3, Theorem 5] *A ternary group isotope* $(Q; f)$ *belongs to* $\mathfrak{P}(D_8)$ *if and only if there exists an abelian group* $(Q, +, 0)$*, its involutive automorphism* α *and an element* $a \in Q$ *such that* $\alpha(a) = -a$ *and*

$$f(x, y, z) = \alpha x + \alpha y - z + a. \tag{5}$$

By Proposition 6 in [3], if $\mathrm{Ps}(f) = S_3$, then different parastrophes are f, ${}^{(14)}f$, ${}^{(24)}f$, ${}^{(34)}f$.

Theorem 8. [3, Theorem 6] *A ternary group isotope* $(Q; f)$ *belongs to* $\mathfrak{P}(S_3)$ *if and only if there exists an abelian group* $(Q, +, 0)$*, its bijection* α *and an element* $a \in Q$ *such that*

$$f(x, y, z) = \alpha x + \alpha y + \alpha z + a. \tag{6}$$

Theorem 8 and Lemma 5 imply the following statement.

Corollary 9. *All distinct parastrophes of a ternary group isotope* $(Q; f)$ *with the parastrophy symmetry group* S_3 *are*

$$\begin{aligned} f(x, y, z) &= \alpha x + \alpha y + \alpha z + a, & {}^{(24)}f(x, y, z) &= \alpha^{-1}(-\alpha x + y - \alpha z - a), \\ {}^{(14)}f(x, y, z) &= \alpha^{-1}(x - \alpha y - \alpha z - a), & {}^{(34)}f(x, y, z) &= \alpha^{-1}(-\alpha x - \alpha y + z - a). \end{aligned} \tag{7}$$

By Proposition 8 in [3], if $\mathrm{Ps}(f) = A_3$, then f has only different parastrophes f, ${}^{(12)}f$, ${}^{(14)}f$, ${}^{(24)}f$, ${}^{(34)}f$, ${}^{(124)}f$, ${}^{(134)}f$, ${}^{(142)}f$. By Theorem 7, a group isotope with the parastrophic symmetry group A_3 has decomposition (6).

3. Main results

Two transformations α and β of a group $(Q;+)$ are supposed to be equivalent if $\alpha = \gamma \cdot \beta$, where γ is a bijection of $(Q;+)$, i.e., equivalent transformations are invertible on $(Q;+)$ simultaneously. If the transformations are given in the determinant form, then to equivalent transformations there corresponds determinants up to common row or column multipliers or to rearrangements of rows or columns.

Group isotopes whose parastrophic symmetry group is D_8. The necessary and sufficient conditions for a dihedrally symmetric group isotope to be a top quasigroup were announced in [10]. Here, we provide the complete proof of this criterion. Later, the corresponding criterion for a linear ternary quasigroup over a unitary, associative, commutative ring was announced without proof in [11] and some of its corollaries were presented.

Theorem 7 and Corollary 4 imply immediately the following statement.

Proposition 10. *A ternary group isotope with the parastrophic symmetry group D_8 is a medial quasigroup.*

Theorem 11. *A ternary group isotope $(Q; f)$ with the group of parastrophic symmetry D_8 is a top quasigroup if and only if it has canonical decomposition (5), where $(Q, +, 0)$ is an abelian group, α is its involutive automorphism, an element $a \in Q$ such that $\alpha(a) = -a$, and $\alpha + \iota$ is an automorphism of $(Q; +)$.*

Proof. Let $(Q; f)$ be a ternary group isotope and $\mathrm{Ps}(f) = D_8$. By item 5 of Theorem 3, ${}^{\sigma}f$ and ${}^{\tau}f$ are different parastrophes if and only if σ and τ belong to different elements of the set

$$S_4/D_8 = \{D_8,\ (14)D_8,\ (24)D_8\}.$$

We may choose ι, (23), (13) as the representatives of S_4/D_8. Hence, all parastrophes of $(Q; f)$ are principal. Consequently, the classes of parastrophic-orthogonal, self-orthogonal and totally parastrophic-orthogonal ternary group isotopes with the group of parastrophic symmetry D_8 coincide.

By Theorem 7, this group isotope has decomposition (5), and by Lemma 5, its distinct parastrophes are

$$\begin{aligned} f(x, y, z) &= \alpha x + \alpha y - z + a, \\ {}^{(23)}f(x, y, z) &= \alpha x - y + \alpha z + a, \\ {}^{(13)}f(x, y, z) &= -x + \alpha y + \alpha z + a. \end{aligned}$$

By item 1 of Lemma 6, the parastrophes f, ${}^{(23)}f$ and ${}^{(13)}f$ are orthogonal if and only if the determinant

$$d_{D_8} = \begin{vmatrix} \alpha & \alpha & -\iota \\ \alpha & -\iota & \alpha \\ -\iota & \alpha & \alpha \end{vmatrix} \tag{8}$$

is an automorphism of the group $(Q; +)$. Adding the first row to the second one multiplying by $-\iota$, then adding the second and third columns, we get the following transformations for d_{D_8}:

$$d_{D_8} = \begin{vmatrix} 0 & \alpha+\iota & -(\alpha+\iota) \\ \alpha & -\iota & \alpha \\ -\iota & \alpha & \alpha \end{vmatrix} = \begin{vmatrix} 0 & 0 & -(\alpha+\iota) \\ \alpha & \alpha-\iota & \alpha \\ -\iota & 2\alpha & \alpha \end{vmatrix} =$$

$$= -(\alpha+\iota)(2\alpha^2 + \alpha - \iota) = -(\alpha+\iota)(2\iota + \alpha - \iota) = (-\iota)(\alpha+\iota)(\alpha+\iota).$$

Since $-\iota$ is invertible, by Lemma 1 the determinant d_{D_8} is invertible if and only if the transformation $\alpha + \iota$ is an automorphism of the group $(Q; +)$. □

Example 12. Let $\mathbb{Z}_{15}$ be a ring of integers modulo 15. By Theorem 11, $(\mathbb{Z}_{15}; f)$, where

$$f(x,y,z) = x + y - z,$$

is a top quasigroup with $\mathrm{Ps}(f) = D_8$, since $\alpha + \iota = 1 + 1 = 2$ is relatively prime to 15 and so is invertible in the ring $\mathbb{Z}_{15}$.

Corollary 13. *A ternary group isotope with the group of parastrophic symmetry D_8 is not a strongly top quasigroup.*

Proof. Suppose that $(Q; f)$ satisfies the conditions of the corollary and is a strongly top quasigroup. By item 2 of Lemma 6, all minors of d_{D_8} which is defined by (8) should be invertible. However, d_{D_8} contain the minor

$$\begin{vmatrix} \alpha & -\iota \\ -\iota & \alpha \end{vmatrix} = \alpha^2 - \iota = \iota - \iota = 0, \tag{9}$$

which is a contradiction. Therefore, a strongly top group isotope with the group of parastrophic symmetry D_8 does not exist. □

Corollary 14. *There does not exist any strongly orthogonal parastrophes of a ternary group isotope with the group of parastrophic symmetry D_8.*

Proof. Suppose that $(Q; f)$ is a quasigroup with (5). Since each two rows of the determinant d_{D_8} contains the minor (9), the pairs of parastrophes $\left({}^{(13)}f, {}^{(23)}f\right)$, $\left(f, {}^{(13)}f\right)$ and $\left(f, {}^{(23)}f\right)$ can not be strongly orthogonal. □

Group isotopes whose parastrophic symmetry group is S_3. Earlier, the necessary and sufficient conditions for a group isotope with the parastrophic symmetry group S_3 to be a top quasigroup were announced in [12]. Here, we provide the complete proof of this criterion and some of its corollaries.

Lemma 15. *A triplet of parastrophes f, ${}^{\tau}f$, ${}^{\nu}f$, where $\tau, \nu \in \{(14), (24), (34)\}$, of a ternary medial quasigroup $(Q; f)$ with the group of parastrophic symmetry S_3 is orthogonal if and only if it has canonical decomposition (6), $a \in Q$, and α, $\alpha + \iota$ are automorphisms of the group $(Q; +)$.*

Proof. Let the conditions of the lemma be true. According to Proposition 2 and item 1 of Lemma 6, the parastrophes f, ${}^{\tau}f$, ${}^{\nu}f$ are orthogonal if and only if the determinant (4) is an automorphism of the group $(Q; +)$, where one of its row is α, α, α and others are two of the following sequences:

$$-\iota, \alpha, \alpha; \qquad \alpha, -\iota, \alpha; \qquad \alpha, \alpha, -\iota.$$

Note that the invertibility property for a determinant is invariant under permutations of its rows and columns.

By permuting the rows, we can put the row α, α, α first. By permuting the columns, we can get $-\iota, \alpha, \alpha$ as the second row. If the third row is $\alpha, \alpha, -\iota$, we permute the second and third columns to obtain the following determinant:

$$d^1_{S_3} = \begin{vmatrix} \alpha & \alpha & \alpha \\ -\iota & \alpha & \alpha \\ \alpha & -\iota & \alpha \end{vmatrix} = \begin{vmatrix} \alpha+\iota & 0 & 0 \\ -\iota & \alpha & \alpha \\ \alpha & -\iota & \alpha \end{vmatrix} = (\alpha+\iota)(\alpha^2+\alpha) = (\alpha+\iota)\alpha(\alpha+\iota). \tag{10}$$

By Lemma 1, the deteminant $d^1_{S_3}$ is invertible if and only if the transformations α and $\alpha+\iota$ are automorphisms of the group $(Q;+)$. □

Lemma 16. *A triplet of parastrophes ${}^{(14)}f$, ${}^{(24)}f$, ${}^{(34)}f$ of a ternary medial quasigroup $(Q;f)$ with the group of parastrophic symmetry S_3 is orthogonal if and only if it has canonical decomposition (6), $a\in Q$, and α, $\alpha+\iota$, $2\alpha-\iota$ are automorphisms of the group $(Q;+)$.*

Proof. Let the conditions of the lemma be true. According to Proposition 2 and item 1 of Lemma 6, the parastrophes ${}^{(14)}f$, ${}^{(24)}f$, ${}^{(34)}f$ are orthogonal if and only if the determinant (4) is an automorphism of the group $(Q;+)$ whose rows are the following sequences:

$$-\iota,\alpha,\alpha;\qquad \alpha,-\iota,\alpha;\qquad \alpha,\alpha,-\iota.$$

Under a permutation of the rows, we obtain the following determinant:

$$d^2_{S_3}=\begin{vmatrix} -\iota & \alpha & \alpha\\ \alpha & -\iota & \alpha\\ \alpha & \alpha & -\iota\end{vmatrix}. \tag{11}$$

Adding the first row to the second one multiplied by $-\iota$, and then adding the first and second columns, results in the following transformations for $d^2_{S_3}$:

$$d^2_{S_3}=\begin{vmatrix} -(\alpha+\iota) & \alpha+\iota & 0\\ \alpha & -\iota & \alpha\\ \alpha & \alpha & -\iota\end{vmatrix}=\begin{vmatrix} -(\alpha+\iota) & 0 & 0\\ \alpha & \alpha-\iota & \alpha\\ \alpha & 2\alpha & -\iota\end{vmatrix}=$$

$$=-(\alpha+\iota)\big(-(\alpha-\iota)-2\alpha^2\big)=-(\alpha+\iota)\big((\iota-\alpha^2)-\alpha(\iota+\alpha)\big)=$$
$$=(-\iota)(\alpha+\iota)(\alpha+\iota)(\iota-2\alpha)=\alpha(\alpha+\iota)(\alpha+\iota)(2\alpha-\iota).$$

By Lemma 1, the deteminant $d^2_{S_3}$ is invertible if and only if the transformations α, $\alpha+\iota$ and $2\alpha-\iota$ are automorphisms of the group $(Q;+)$. □

Theorem 17. *A ternary medial quasigroup $(Q;f)$ with the group of parastrophic symmetry S_3 is a top quasigroup if and only if it has canonical decomposition (6), $a\in Q$, and α, $\alpha+\iota$, $2\alpha-\iota$ are automorphisms of the group $(Q;+)$.*

Proof. Suppose that $(Q;f)$ is a ternary medial quasigroup and $\mathrm{Ps}(f)=S_3$. By item 5 of Theorem 3, the parastrophes ${}^{\sigma}f$ and ${}^{\tau}f$ are different if and only if σ and τ belong to different elements of the set

$$S_4/S_3=\{S_3,\ (14)S_3,\ (24)S_3,\ (34)S_3\}.$$

In other words, all pairwise different parastrophes are f, ${}^{(14)}f$, ${}^{(24)}f$, ${}^{(34)}f$.

Consequently, if the parastrophic symmetry group of a ternary quasigroup is S_3, then all its principal parastrophes coincide. By Theorem 8, this group isotope has decomposition (6), and by Corollary 9, its distinct parastrophes are (7).

Thus, the proof of the theorem follows from Lemma 15 and Lemma 16. □

Example 18. Let $\mathbb{Z}_{21}$ be a ring of integers modulo 21. By Theorem 17, $(\mathbb{Z}_{21};f)$, where

$$f(x,y,z)=10x+10y+10z, \tag{12}$$

is a top quasigroup with $\mathrm{Ps}(f)=S_3$, since

$$\alpha+\iota=10+1=11,\qquad 2\alpha-\iota=2\cdot 10-1=19.$$

We may formulate some generalizations for a cyclic quasigroup $(\mathbb{Z}_m; f)$ with (12) as follows:

Corollary 19. *Let $\mathbb{Z}_m$ be a ring of integers modulo m, and the operation f be defined by (12).*

1. *$(\mathbb{Z}_m; f)$ is a top quasigroup with $\mathrm{Ps}(f) = S_3$ if and only if m is relatively prime to 2, 3, 5, 7, 11 and 19.*
2. *If $m = p$ is a prime number, then $(\mathbb{Z}_p; f)$ is a top quasigroup with $\mathrm{Ps}(f) = S_3$ for each prime $p > 19$.*
3. *If p is the least prime divisor of m, then $(\mathbb{Z}_m; f)$ is a top quasigroup with $\mathrm{Ps}(f) = S_3$ for each prime $p > 19$.*

Corollary 20. *A triplet of parastrophes ${}^{\sigma}f$, ${}^{\tau}f$, ${}^{\nu}f$ of a medial quasigroup $(Q; f)$ with the group of parastrophic symmetry S_3 is strongly orthogonal if and only if $\{\sigma, \tau, \nu\} = \{(14), (24), (34)\}$, f has canonical decomposition (6), and α, $\alpha + \iota$, $2\alpha - \iota$, $\alpha - \iota$ are automorphisms of $(Q; +)$.*

Proof. By item 2 of Lemma 6, we should consider all non-trivial minors of the determinants $d^1_{S_3}$ and $d^2_{S_3}$ defined by (10) and (11) respectively. The determinant (10) contains the minor

$$\begin{vmatrix} \alpha & \alpha \\ \alpha & \alpha \end{vmatrix} = 0.$$

Therefore, the triplet ${}^{\sigma}f$, ${}^{\tau}f$, ${}^{\nu}f$ does not contain the operation f and so

$$\{\sigma, \tau, \nu\} = \{(14), (24), (34)\}.$$

Consider the determinant (11) and its nine minors. All its minors are equivalent to two of them under permutations of the rows and columns:

$$\begin{vmatrix} -\iota & \alpha \\ \alpha & -\iota \end{vmatrix} = \iota - \alpha^2 = -(\alpha - \iota)(\alpha + \iota), \qquad \begin{vmatrix} \alpha & \alpha \\ \alpha & -\iota \end{vmatrix} = -\alpha - \alpha^2 = -\alpha(\iota + \alpha).$$

These minors are invertible if and only if $\alpha + \iota$ and $\alpha - \iota$ are automorphisms of $(Q; +)$. Consequently, ${}^{(14)}f$, ${}^{(24)}f$, ${}^{(34)}f$ are strongly orthogonal if and only if α, $\alpha + \iota$, $2\alpha - \iota$ and $\alpha - \iota$ are automorphisms of $(Q; +)$.

□

Example 21. Consider the field $\mathbb{Z}_{13}$ of integers modulo 13. By Corollary 20, $(\mathbb{Z}_{13}; f)$, where

$$f(x, y, z) := 8x + 8y + 8z,$$

is a quasigroup with $\mathrm{Ps}(f) = S_3$ which has strongly orthogonal parastrophes ${}^{(14)}f$, ${}^{(24)}f$, ${}^{(34)}f$, since

$$\alpha + \iota = 8 + 1 = 9, \qquad 2\alpha - \iota = 2 \cdot 8 - 1 = 15, \qquad \alpha - \iota = 8 - 1 = 7.$$

Corollary 22. *Let $(Q; f)$ be a medial quasigroup with (6) and possess the group of parastrophic symmetry S_3. Then*

1. *$(Q; f)$ is not a strongly top quasigroup;*
2. *${}^{(14)}f$, ${}^{(24)}f$ and ${}^{(34)}f$ are orthogonal if and only if $(Q; f)$ is a top quasigroup;*
3. *$(Q; f)$ is parastrophic-orthogonal if and only if $\alpha + \iota$ is an automorphism of $(Q; +)$.*

Proof. Corollary 20 implies immediately item 1. Item 2 follows from Lemma 16 and Theorem 17. Item 3 follows from Lemma 15 and Lemma 16. □

A group isotope $(Q; f)$ with the parastrophic symmetry group S_3 has no principal parastrophes except f, however each of its divisions has three principal parastrophes.

Proposition 23. *Let $(Q; f)$ be a medial quasigroup with (6) and $\mathrm{Ps}(f) = S_3$. For each $i \in \{1, 2, 3\}$, its parastrophe $(Q; {}^{(i4)}f)$ is*

(1) *self-orthogonal if and only if $\alpha + \iota$ and $2\alpha - \iota$ are automorphisms of $(Q; +)$;*
(2) *strongly self-orthogonal if and only if $\alpha + \iota$, $2\alpha - \iota$ and $\alpha - \iota$ are automorphisms of $(Q; +)$.*

Proof. Consider a division of f, say ${}^{(14)}f$, as a principal operation. Then obviously, it has the parastrophic symmetry group $H := \{\iota, (23), (24), (34), (234), (243)\} \leqslant S_4$ which is a conjugate of S_3 by the permutation (14). In this case, $S_4/H = \{H, (12)H, (13)H, (14)H\}$. The operation ${}^{(14)}f$ has three principal parastrophes ${}^{(14)}f$, ${}^{(24)}f$, ${}^{(34)}f$ whose orthogonality and strong orthogonality follow from Lemma 16 and Corollary 20 respectively. □

Note that a ternary quasigroup with the parastrophic symmetry group A_3 exists, as follows from the paper [2] (see for example Theorem 3.3). However, there are no group isotopes with eight distinct parastrophes.

Proposition 24. *If the parastrophic symmetry group of a group isotope contains A_3, then it containes S_3.*

Proof. Theorem 7 from [3] states that a group isotope $(Q; f)$ with $\mathrm{Ps}(f) \supseteq A_3$ has canonical decomposition (6). This implies the equalities

$$ {}^{(12)}f = f, \qquad {}^{(13)}f = f $$

and hence $(12), (13) \in \mathrm{Ps}(f)$. Since the permutations (12), (13) generate S_3, it follows that $S_3 \subseteq \mathrm{Ps}(f)$. □

Conclusions. The necessary and sufficient conditions for a ternary medial quasigroup to be a top quasigroup are given in the cases when the quasigroup has the parastrophic symmetry group D_8 and S_3 (Theorem 11 and Theorem 17). Consequently, this provide methods of constructing ternary orthogonal quasigroups which have 3 and 4 distinct orthogonal parastrophes. Besides, we have shown that a medial quasigroup with $\mathrm{Ps}(f) = S_3$ may have a triplet of strongly orthogonal parastrophes (Corollary 20). A method of constructing a triplet of ternary strongly parastrophic-orthogonal quasigroups follows.

From the obtained results, we have the following theorem.

Theorem 25. *Let $(Q; +)$ be an abelian group and φ be its automorphism. Then the operations f_1, f_2, f_3 defined by*

$$ f_1(x, y, z) = \varphi x + y + z, \qquad f_2(x, y, z) = x + \varphi y + z, \qquad f_3(x, y, z) = x + y + \varphi z $$

are strongly orthogonal quasigroup operations if and only if φ, $\varphi - 2\iota$, $\varphi - \iota$, $\varphi + \iota$ are automorphisms of $(Q; +)$.

Proof. Let the conditions of the theorem hold. Then the quasigroup $(Q; f)$ defined by

$$ f(x, y, z) = \alpha x + \alpha y + \alpha z $$

is medial. According to Corollary 9, all divisions of this quasigroup can be expressed in the form

$$ {}^{(14)}f(x, y, z) = I(-\alpha^{-1}x + y + z), \qquad {}^{(24)}f(x, y, z) = I(x - \alpha^{-1}y + z), $$
$$ {}^{(34)}f(x, y, z) = I(x + y - \alpha^{-1}z), $$

where $I(x) := -x$. By Corollary 20, these operations are strongly orthogonal if and only if α, $2\alpha-\iota$, $\alpha+\iota$, $\alpha-\iota$ are automorphisms of $(Q;+)$. Let $\varphi := I\alpha^{-1}$. Then $\alpha := I\varphi^{-1}$, and the given conditions mean that

$$I\varphi^{-1},\quad 2I\varphi^{-1}-\iota = I\varphi^{-1}(\varphi-2\iota),\quad I\varphi^{-1}+\iota = \varphi^{-1}(\varphi-\iota),\quad I\varphi^{-1}-\iota = I\varphi^{-1}(\varphi+\iota)$$

are automorphisms of $(Q;+)$. This proves the theorem. □

Corollary 26. *Let $\mathbb{Z}_m$ be a ring of integers modulo m. Then the operations f_1, f_2, f_3 defined by*

$$f_1(x,y,z) = kx+y+z,\qquad f_2(x,y,z) = x+ky+z,\qquad f_3(x,y,z) = x+y+kz$$

are strongly orthogonal quasigroup operations if and only if k, $k-2$, $k-1$, $k+1$ are relatively prime to m.

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References

1. McLeish, M. (1979). *On the number of conjugates of n-ary quasigroups*, Can. J. Math. **XXXI** (3), 637–654. https://doi.org/10.4153/CJM-1979-064-6
2. McLeish M. (1980). *On the Existence of Ternary Quasi-Groups with Two or Eight Conjugacy Classes*, Journal of Combinatorial Theory, Series A, **29**, 199–211. https://doi.org/10.1016/0097-3165(80)90009-6
3. Sokhatsky, F., Pirus, Ye. (2018). *Classification of ternary quasigroups according to their parastrophic symmetry groups, I*, Visnyk DonNu. Series A: Natural Sciences, **1-2**, 70–82. https://doi.org/10.31558/1817-2237.2018.1-2.5
4. Pirus, Ye. (2019). *Classification of ternary quasigroups according to their parastrophic symmetry groups, II*, Visnyk DonNu. Series A: Natural Sciences, **1-2**, 66–75. https://doi.org/10.31558/1817-2237.2019.1-2.9
5. Fryz I., Sokhatsky, F. (2022). *Construction of medial ternary self-orthogonal quasigroups*, Bul. Acad. Stiinte Repub. Mold. Mat., **3** (100), 41–55. https://doi.org/10.56415/basm.y2022.i3.p41
6. Belyavskaya, G.B., Popovich, T.V. (2010). *Totally conjugate orthogonal quasigroups and complete graphs*, J. Math. Sci., **185** (2), 184–191. https://doi.org/10.1007/s10958-012-0907-z
7. Sokhatsky, F. (2016). *Parastrophic symmetry in quasigroup theory*, Visnyk DonNu. Series A: Natural Sciences, **1-2**, 70–83.
8. Sokhatsky, F. (2017). *Factorization of operations of medial and abelian algebras*, Visnyk DonNY. Series A: Natural Sciences, **1-2**, 84–96. https://doi.org/10.31558/1817-2237.2017.1-2.7
9. Belyavskaya, G., Mullen, G.L. (2006). *Strongly orthogonal and uniformly orthogonal many-placed operations*, Algebra Discrete Math., **5**, 1, 1–17.
10. Pirus, Ye. (2020). On ternary top-quasigroups whose group of parastrophic symmetry is D_8. *Book of Abstracts of International mathematical conference dedicated to the 60th anniversary of the department of algebra and mathematical logic of Taras Shevchenko National University of Kyiv* (Kyiv, Ukraine, July 14–17, 2020), 64.
11. Rotari, T. (2024). On ternary quasigoups with exactly three distinct and orthogonal parastrophes. *e-Book of Abstracts of the 31nd International Conference on Applied and Industrial Mathematics* (Oradea, Romania, September 19–22, 2024), 53–54.
12. Pirus, Ie. (2015). About parastrophic orthogonality of medial ternary quasigroups. *Abstracts of X International Algebraic Conference in Ukraine dedicated to the 70th anniversary of Yu.A. Drozd* (Odesa, Ukraine, August 20–27, 2015), 87.

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Парастрофно-ортогональні тернарні медіальні квазігрупи, які мають 3 і 4 різних парастрофи

Ірина Фриз, Євген Пірус

Анотація. У цій статті ми вивчаємо парастрофно-ортогональні тернарні квазігрупи, а саме, ізотопи груп, які мають 3 і 4 різних парастрофи. Виведено необхідні і достатні умови коли тернарні медіальні квазігрупи, що мають 3 і 4 різних парастрофи, є тотально парастрофно-ортогональними. Описано за яких умов такі квазігрупи є строго парастрофно-ортогональними. Таким чином, отримано деякі методи побудови ортогональних і строго-ортогональних квазігруп.

Ключові слова: тернарна квазігрупа, ізотоп групи, медіальна квазігрупа, парастроф, (сильно) ортогональні квазігрупи, тотально парастрофно-ортогональна (top) квазігрупа.

Список використаних джерел

1. McLeish M. On the number of conjugates of n-ary quasigroups. *Can. J. Math.* 1979. Vol. XXXI, № 3. P. 637–654. DOI: https://doi.org/10.4153/CJM-1979-064-6
2. McLeish M. On the Existence of Ternary Quasi-Groups with Two or Eight Conjugacy Classes. *Journal of Combinatorial Theory, Series A.* 1980. Vol. 29, Iss. 2. P. 199–211. DOI: https://doi.org/10.1016/0097-3165(80)90009-6
3. Sokhatsky F., Pirus Ye. Classification of ternary quasigroups according to their parastrophic symmetry groups, I. *Visnyk DonNu. Series A: Natural Sciences.* 2018. № 1-2. P. 70–82. DOI: https://doi.org/10.31558/1817-2237.2018.1-2.5
4. Pirus Ye. Classification of ternary quasigroups according to their parastrophic symmetry groups, II. *Visnyk DonNu. Series A: Natural Sciences.* 2019. № 1-2. P. 66–75. DOI: https://doi.org/10.31558/1817-2237.2019.1-2.9
5. Fryz I., Sokhatsky F. Construction of medial ternary self-orthogonal quasigroups. *Bul. Acad. Stiinte Repub. Mold. Mat.* 2022. № 3(100). P. 41–55. DOI: https://doi.org/10.56415/basm.y2022.i3.p41
6. Belyavskaya G.B., Popovich T.V. Totally conjugate orthogonal quasigroups and complete graphs. *J. Math. Sci.* 2012. Vol. 185, No. 2. P. 184–191. DOI: https://doi.org/10.1007/s10958-012-0907-z
7. Sokhatsky F. Parastrophic symmetry in quasigroup theory. *Visnyk DonNu. Series A: Natural Sciences.* 2016. № 1-2. P. 70–83.
8. Sokhatsky F. Factorization of operations of medial and abelian algebras, *Visnyk DonNY. Series A: Natural Sciences.* 2017. № 1-2. P. 84–96. DOI: https://doi.org/10.31558/1817-2237.2017.1-2.7
9. Belyavskaya G., Mullen G.L. Strongly orthogonal and uniformly orthogonal many-placed operations. *Algebra Discrete Math.* 2006. Vol. 5, № 1. P. 1–17.
10. Pirus Ye. On ternary top-quasigroups whose group of parastrophic symmetry is D_8. *International mathematical conference dedicated to the 60th anniversary of the department of algebra and mathematical logic of Taras Shevchenko National University of Kyiv:* Book of Abstracts (Kyiv, July 14–17, 2020) / Taras Shevchenko National University of Kyiv, 2020. P. 64.
11. Rotari T. On ternary quasigoups with exactly three distinct and orthogonal parastrophes. *The 31nd International Conference on Applied and Industrial Mathematics:* e-Book of Abstracts (Oradea, Romania, September 19–22, 2024) / Romanian Society of Applied and Industrial Mathematics. P. 53–54.
12. Pirus Ie. About parastrophic orthogonality of medial ternary quasigroups. *X International Algebraic Conference in Ukraine dedicated to the 70th anniversary of Yu.A. Drozd:* Abstracts (Odessa, August 20–27, 2015) / I.I. Mechnikov Odessa National University. P. 87.

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